Space Elevator Power System Analysis and Optimization

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Abstract

This paper lays out the basics constraints for a Space Elevator power system, performs parameter optimization, and compares the results with real-life technology parameters. The paper also considers the special case of solar climbers that have the additional constraint of a once-per-day launch rate.

1 Motivation

The Space Elevator is a transportation system, and as these systems go, we're interested in moving as much payload as possible to orbit for the lowest overall cost.

Because the equations governing the Space Elevator are simple, it is possible to make quite a bit of headway using simple mathematical analysis.

Since the Space Elevator is linearly scalable, we normalize the calculations by the maximum mass that is allowed to hang from the bottom of the tether (m_{max}). Thus a "20-ton" Elevator is one that can support a single 20 ton climber at ground level. Typically, this means that the tether weight 4000 – 6000 tons, and the climbers will actually weigh around 15 tons (since we have multiple climbers simultaneously on the ribbon). Using m_{max} normalized mass units, the tether weighs 200-300, the climber weighs 0.75, etc. We'll define a "Standard throughput unit" (STU) as being able to launch one m_{max} per year. Unless specified otherwise, we'll be talking about the payload throughput.

To simplify matters, we divide the mass of the climber into payload and power system, assuming the "dead structure" is small in comparison to either of them. (Any structure that scales with the power system (e.g. motors) is incorporated into the overall power density of the power system).

The parameters determined by technology are the power density of the power system, and the maximum speed we can handle the tether. The variables we can tune as part of the design are the climber mass, the power-payload ratio division within the climber, and the time between climber launches.

2 Ascent power profile

The power required to move a mass at a certain velocity is a function of the effective gravity at the altitude that the climber is at: $P = m \cdot g_{eff} \cdot v$, where $g_{eff}(r) = g(r_e/r)^2 - \omega^2 rm$ (r_e=6400 km is the radius of the earth, and ω_e = 7.3E-5 rad/sec, the angular velocity of the Earth).

For $r < 2r_e$, we can approximate this very well as simply $g_{eff}(r) = g(r_e/r)^2$, which means that for a specific power system, the climber's velocity will increase according to a square law $v(r) = P/(m \cdot g) \cdot (r/r_e)^2$ as it moves out, until it reaches some maximum terminal velocity v_T determined by the tether handling system. From that point onwards, the climber moves at v_T, and the power system is under-utilized.

For a climber with power density (mass per power) ρ_P and power system mass fraction β, the total available power is $P = m \cdot \beta \cdot \rho_P$, the initial velocity is $v_e = \beta \cdot \rho_P / g$, and so the velocity is $v(r) = \min[V_T, v_e \cdot (r/r_e)^2]$. The terminal velocity point r_T is where the climber reaches v_T, and that happens at $v_T = v_e \cdot (r_T/r_e)^2$, or at $r_T = r_e(v_T/v_E)^{0.5}$. The amount of payload carried by the climber is of course $(1-\beta)m_{max}$.

The formula for the time it takes a climber that is following a constant power velocity profile to cover the distance between r_e and r is:

$$t = \int_{x=r_e}^{x=r} dx / v(x) = \int_{x=r_e}^{x=r} v_e^{-1} (x/r_e)^{-2} dx = (r_e/r)(r - r_e)/v_e = t_0 / Q$$

Where $t_0 = (r - r_e)/v_e$ is the time it would have taken the climber to cover the distance if it were moving at a constant velocity, and $Q = r/r_e$ is the radius ratio.

The distance traveled by a constant power velocity climber (relative to re) is: $d/r_e = (r-r_e)/r_e = t/(r_e/v_e - t)$.

3 Handoff

The spacing between climbers can be characterized by a handoff fraction k_H, so that a new climber is launched when the old climber reached $g_{eff}/g=k_H$. The handoff altitude $r_H = r_e/k_H^{0.5}$ is the location where this happens, and the time since launch that this happens at is t_H)

The mass of each climber can only be $(1-k_H)m_{max}$, so that the geometrical series $(1-k_H)+(1-k_H)k_H+(1-k_H)k_H^2\ldots =1$. (This is a slightly conservative, since the spacing between the climbers does not remain constant, and there are only a finite number of climbers).

If the initial parameters are such that $r_T>r_H$, (or $(v_T/v_e)^{0.5} > k_H^{-0.5}$) then the climber will follow a constant-power velocity profile all the way out to the handoff point. We call this a "power limited" profile. Otherwise, the climber will "max-out" on the way to the handoff point, and the profile is called "speed limited". There is also the possibility the $r_T < r_e$, which means that the climber power system starts out under-utilized, even at ground level. This is clearly a non-optimal case.

4 Throughput

The payload per climber is therefore $m_P= (1-\beta)(1-k_H)$ and the mass throughput of the system is $P=(1-\beta)(1-k_H)/t_H$. The more frequently we launch climbers, the smaller each one can be, but the larger the throughput. This trend continues to the limiting case of a continuous (variable speed) belt of cargo, though we see no practical way of doing that. Similarly, the faster we move, the more climbers we can launch, but the larger power systems leave less room for payload.

The parameters dictated by technology are the power density ρ_P and the terminal velocity v_T. The variables we can tune are the handoff constant k_H, the power system mass fraction β, and the handoff time t_H. The variables represent only 2 degrees of freedom, since clearly once we set β and k_H, t_H is already determined.

When power limited scenarios, while v_T can still be tuned, it does not affect the throughput, since it only alters the behavior of the climber beyond the handoff point.

5 Optimization

To find the maximum throughput:

The relative payload is $m_{payload} = (1-k_H)(1-\beta)$
And the throughput is $P = (1-k_H)(1-\beta)/t_H$

We can now link β and k_H through t_H

Speed-limited:

Initial velocity:	$v_e = \beta\cdot\rho_P/g$
Terminal point:	$r_T = r_e(v_T/v_e)^{0.5} = r_e\cdot Q_T$
Terminal altitude:	$a_T = r_T-r_e$
Time to terminal point:	$t_T = (r_e/r_T)(r_T-r_e)/v_e = (r_e/v_e)(Q_T-1)/Q_T$
Handoff point:	$r_H = r_e/k_H^{0.5}$
Handoff altitude:	$a_H = r_H-r_e$
Time to handoff point:	$t_H = t_T + (r_H-r_T)/v_T = (r_e/v_e)(Q_T-1)/Q_T + (r_ek^{-0.5}-r_eQ_T)/v_T =$ $(r_e/v_T)\cdot[Q_T(Q_T-1)-Q_T + k_H^{-0.5}] = (r_e/v_T)[Q_T(Q_T-2) + k_H^{-0.5}]$
Extracting k_H:	$k_H = [(t_Hv_T/r_e) - Q_T(Q_T-2)]^{-2}$

Power-limited:

Initial velocity:	$v_e = \beta\cdot\rho_P/g$
Handoff point:	$r_H = r_e(v_H/v_e)^{0.5} = r_e\cdot Q_H = r_e\cdot(k_H)^{-0.5}$
Handoff altitude:	$a_H = r_H-r_e$
Velocity at handoff point:	$v_H = v_eQ_H^2$
Time to handoff point:	$t_H = (r_e/r_H)(r_H-r_e)/v_e = (r_e/v_e)(Q_H-1)/Q_H = (r_e/v_H)Q_H(Q_H-1)$
Extracting β:	$\beta = g\cdot v_e/\rho_P = (g/\rho_P)\cdot(r_e/t_H)\cdot(Q_H-1)/Q_H = c_H\cdot(Q_H-1)/Q_H$
Solving for k_H	$k_H = Q_H^{-2} = [c_H / (c_H-\beta)]^{-2}$

It is possible to express $dP/d\beta$ is closed form, but the resulting expression is only solvable numerically. For the same effort, it is more interesting to directly optimize $P(\beta)$.

6 Results

The worksheet is implemented in MS Excel, and works with the built-in numerical solver to yield optimal values for P in respect to β. Below is one instance of the optimization, for ρ_P = 1500 and v_T = 80. The source worksheet is available online.

Earth radius	r_e	m	6.4E6	6.4E6	6.4E6	6.4E6	6.4E6	6.4E6	r_e	
Earth rotation	ω_e	rad/sec	7.3E-5	7.3E-5	7.3E-5	7.3E-5	7.3E-5	7.3E-5	ω_e	
Earth gravity	g	m/sec2	9.8E0	9.8E0	9.8E0	9.8E0	9.8E0	9.8E0	g	
power density	ρ_P	watt/kg	1500	1500	1500	1500	1500	1500	ρ_P	
terminal velocity	v_T	m/s	80	80	80	80	80	80	v_T	
delta-β		0.02	-0.04	-0.02	0	0.02	0.04			
power mass ratio	β		0.199	0.219	0.239	0.259	0.279	0.239	β	[1]
handoff time	t_H	sec	86400	86400	86400	86400	86400	86400	t_H	
Initial Velocity	v_e	m/s	30.5	33.6	36.6	39.7	42.7	36.6	v_e	[5]

sqrt(v_T/v_e)	Q_T		1.6	1.5	1.5	1.4	1.4	1.5	Q	
terminal point	r_T	m	1.0E7	9.9E6	9.5E6	9.1E6	8.8E6	9.5E6	r_T	
terminal altitude	a_T	km	3966	3482	3060	2688	2356	3060	a_T	[7]
terminal time	t_T	sec	8.0E4	6.7E4	5.7E4	4.8E4	4.0E4	5.7E4	t_T	
		hr	22.3	18.7	15.7	13.3	11.2	15.7		

handoff constant	k_H		0.348	0.314	0.292	0.276	0.264	0.292	k_H	[4]
handoff point	r_H	m	1.1E7	1.1E7	1.2E7	1.2E7	1.2E7	1.2E7	r_H	
handoff altitude	a_H	km	4454	5018	5449	5783	6045	5449	a_H	[6]
handoff time (chk)	t_H	sec	86400	86400	86400	86400	86400	86400	t_H	
		hr	24	24	24	24	24	24		
$m_{payload}$	m_p	m_{max}	0.52	0.54	0.54	0.54	0.53	0.539	m_p	[3]
Throughput	P	STU	191	195	197	196	194	197	P	[2]
			1	1	1	1	1	1		

Ignore columns with red indicators

			0	0	0	0	0	0		
	C_H		0.484	0.484	0.484	0.484	0.484	0.484		
	Q_H		1.700	1.828	1.978	2.154	2.364	1.978		
handoff constant	k_H		0.346	0.299	0.256	0.216	0.179	0.256	k_H	[4]
handoff point	r_H	m	1.1E7	1.2E7	1.3E7	1.4E7	1.5E7	1.3E7	r_H	
handoff altitude	a_H	km	4478	5300	6256	7383	8729	6256	a_H	[6]
handoff time (chk)	t_H	sec	86400	86400	86400	86400	86400	86400	t_H	
		hr	24	24	24	24	24	24		
$m_{payload}$	m_p	m_{max}	0.524	0.547	0.566	0.581	0.592	0.566	m_p	[3]
Throughput	P	STU	191	200	207	212	216	207	P	[2]
Throughput	P	STU	191	195	197	196	194	197	P	[2]

Instructions:
Enter values in the 3 parameter cells marked:
Experiment with the value in the β cell:
Control the 5 test case columns using delta-β:

The Red/Green indicators show which scenario (power or speed limited) is applicable.
(Red-Red means a different condition (such as v_e > v_T) is violated)

When ready, hit alt-T,V (tools-->solver) and optimize P with respect to β.

The parameter space is 3-dimensional, and we are interested in quite a few of the resulting quantities. The approach taken for aggregating the data is to hold t_H constant and plot one table per observed quantity, then experiment with other values of t_H.

This process requires a considerable amount of manual work, but gives the experimenter a good insight into the behavior of the system.

Below are the results for daily-cycle operations (t_H=86400). Power limited scenarios are shaded. Our focus is on the payload throughput (P). m_P and k_H are shown for "situational awareness".

[1] β	500	700	1000	1500	2500	3500
30	0.374					
40	0.420					
60	0.450	0.381	0.295	0.220	0.150	
80	0.450	0.415	0.321	0.239	0.165	
100	0.450	0.426	0.340	0.253	0.174	0.136
120	0.450	0.426	0.355	0.263	0.180	0.141

[2] P	500	700	1000	1500	2500	3500
30	101					
40	105					
60	105	135	162	186	209	
80	105	137	168	197	224	
100	105	137	172	203	233	248
120	105	137	173	208	240	256

[3] m_p	500	700	1000	1500	2500	3500
30	0.278					
40	0.287					
60	0.288	0.369	0.443	0.509	0.571	
80	0.288	0.374	0.461	0.539	0.612	
100	0.288	0.375	0.470	0.557	0.639	0.679
120	0.288	0.375	0.475	0.569	0.657	0.700

[4] k_H	500	700	1000	1500	2500	3500
30	0.557					
40	0.506					
60	0.477	0.403	0.372	0.347	0.328	
80	0.477	0.360	0.322	0.292	0.267	
100	0.477	0.347	0.288	0.255	0.227	0.214
120	0.477	0.347	0.264	0.229	0.199	0.185

The first observation is that once the system becomes power-limited, v_T (as expected) no longer influences the result. The second observation is that even before the system becomes power-limited, the performance only advances slowly. If we stay in the "reasonable" v_T range of 60-120 m/s, the throughput values are mostly a function of the power density.

As estimated before (using the r_H=2r_e point), the weaker power systems run with $m_P \approx 0.25$, but we find out that the stronger ones reach much higher, into $m_P \approx 0.6$. Since even with $m_P = 0.6$ we only have P = 219, let's look at what can be gained by increasing the launch rates.

Looking at bi-daily operations (t_H=43200) and keeping in mind that for the same P, m_P will be half its previous value, we get:

[1] β	500	700	1000	1500	2500	3500
30	0.440					
40	0.477	0.400				
60	0.477	0.466				
80	0.477	0.466	0.420			
100	0.477	0.466	0.450	0.342	0.235	0.183
120	0.477	0.466	0.450	0.363	0.251	0.196

[2] P	500	700	1000	1500	2500	3500
30	114					
40	115	151				
60	115	155				
80	115	155	209			
100	115	155	210	275	338	369
120	115	155	210	281	352	387

The shaded region is larger since the smaller climbers need to get out of the way faster and so carry larger power systems, thus maxing out sooner. For this reason while the higher performing system gain up to 50% in throughput, the lower performing systems gain only about 10%. This is to be expected, since there's little point expediting the launch rate if the system is not capable of getting the climbers far enough out of the way by in half a day.

7 Conclusions

We can draw the following table, to be used as a rough guide for the throughput available from a power system: (Throughput again is in units of m_{max}/yr, or STU)

	500	**700**	**1000**	**1500**	**2500**	**3500**
Daily	100	135	170	200	230	250
Bi-Daily	115	155	210	275	340	370

When compared to the requirement 200-250 STU imposed by the Space Elevator Feasibility condition for 30 MYuri tethers, these results show very little salvation for the low power density systems, since their performance can't be improved by either a higher v_T, or a lower t_H. For higher performing systems, as long as we remain with daily operations, even with the benefit of high power density systems and faster travel speeds, throughput remains just under 300 STU.

If we need to get to the 300-400 STU throughput range, the only way to get there is to have a 2500 – 3500 kWatt/kg power system, be able to travel at v_T>100 m/s, and get into Bi-Daily operational tempo. Faster-than-daily launches, however, open an entire Pandora's Box of operational issues pertaining to the day-night cycle and power beaming, very fast atmospheric crossings, etc.

In summary:
- The universe has again conspired to make the Space Elevator feasible, but make us work very hard at it.
- High intensity power beaming which requires cooling radiators is probably not a viable power source.
- 1-sun thin film solar technology is adequate from a power density perspective.
- 1-2 sun power beaming is viable and might be needed to augment solar operations.

8 References

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